Dynamic pricing for passenger groups of high-speed rail transportation

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A B S T R A C T

The widespread high-speed rail has become an important transportation mode for tourists in China, who travel primarily in groups. With more attractive price, railway operators may be able to expand ridership, improve customer satisfaction, and ultimately increase revenue. The traditional pricing method for passenger groups, however, is relatively simple, inflexible, which may lead to profit loss during off-seasons. In order to assist railway corporations in revenue improvement, a dynamic pricing model is provided to determine the optimal price for passenger groups. Numerical experiments show that the proposed method is able to suggest the optimal price for total revenue maximization, subject to the capacity constraint.

1. Introduction

High-speed rail (HSR) is gradually becoming the main thrust of railway passenger transportation for long distance travel with the advantage of comfort, speed and security. In China, there are 2673.5 pairs of general trains and 1556.5 pairs of electrical multiple units (EMU) trains (including the high-speed trains), which can be divided into one-stop HSR and multi-stop HSR. However, HSR is so costly that is difficult to make profit (Ryder, 2012). Recently, China’s plans for HSR construction have slowed down; in the meantime, more attention is given to service improvement and pricing strategy in order to increase ridership and revenue.

Many people like to travel to the same destinations in groups for the same purpose, such as business or sightseeing (McGill and van Ryzin, 1999; Kuyumcu and Garcia-Diaz, 2000; Weatherford, 1997; Rui, 2014). As air passenger transportation, the rail passenger transportation includes individual passengers and group passengers. For the Chinese rail corporation, the railway regulation defines that for passengers with more than 20 persons may be used as group passengers and enjoy discount fare. For JR Group (Japan Rail Group), groups of eight or more adults enjoy 10–15 percent discount, and groups of eight or more students enjoy 30–50 percent discount. Rail operators set discount policies for groups to improve their revenue. The price of the group tickets is an important factor influencing the aggregate revenue from the sales of all tickets.

Dynamic pricing and seat inventory control are two main ingredients of revenue management. In particular, dynamic pricing includes setting up multiple fares in advance and then dynamically varying the availability of each according to...
demand forecasts and actual bookings to maximize revenue. On the other hand, seat inventory control is to accept or reject the request of customers at a time for a certain type of product or service in a predetermined period. Seat inventory control decides which class seats to keep open for purchase or to close as the remaining capacity of the fare class is depleted over time with the customer purchases.

In this paper, we propose a revenue management model that uses the dynamic pricing with seat inventory control to handle groups as well as individual travelers. Given an initial seat inventory of a HSR train and a finite time horizon over which sales are allowed, we formulate the tactical problem of dynamically pricing to maximize the aggregate revenue. Our approach consists of a two-stage dynamic program considering groups and individuals, separately. The demand for each fare class is uncertain and it depends on the set price. The booking horizon is finite and divided into multiple time periods. At the beginning of each period, the operator decides how many seats in each fare class are to be provided and at which price to charge in order to maximize the expected revenue.

Pricing is one of key transportation marketing activities. Price affects demand. If demand increases and supply remains unchanged, then it leads to higher ticket price. If demand decreases and supply remains unchanged, then it leads to lower ticket price. The total revenue is the price of the ticket multiplied by the quantity sold. So, comparing with full fare, the lower fare (discount ticket) may lead to higher demand and total revenue during off-season (lower demand season). For boom season (higher demand season), railway operators only provide full fare ticket for groups to maximize his revenue. Our results show that there is an optimal group discount for a specified demand. It is helpful for railway operator to adopt this discount for groups.

The remainder of this paper is structured as follows. Section 2 reviews the earlier literature, summarizing the relevant background to the present study. Section 3 specifies the proposed model. Properties are proofed in Appendix A. The empirical case study is presented in Section 4, with Section 5 concluding the paper by summarizing the present study, and a brief discussion of future research directions to extend this line of enquiry.

2. Literature review

Since little work on dynamic pricing for HSR has been reported due to limited operation experience in recent years, we studied the relevant literature in airline revenue management.

The primary method of airline revenue management is seat inventory control at a given set of prices, and on this subject McGill and van Ryzin made an extensive literature review (McGill and van Ryzin, 1999). Nevertheless, McGill and van Ryzin only reviewed the problems without considering different pricing for group and individual passengers. Kuyumcu and Garcia-Diaz further developed an analytical procedure for a joint pricing and seat allocation problem by considering the number of fare classes and aircraft capacities (Kuyumcu and Garcia-Diaz, 2000); similarly, Weatherford investigated the joint pricing and inventory control problem for a single-leg flight with prices of the different classes (Weatherford, 1997). Both of them solved the mathematical problem by using 0–1 integer programming model which neglected the dynamic booking process and the stochastic nature of the travel needs. In view of the stochastic travel demand, Gallego and van Ryzin looked into a dynamic pricing problem by treating ticket booking as a stochastic process to achieve the objective of maximizing the expected revenue (Gallego and van Ryzin, 1994), and they found the optimal pricing policy in a closed form for a family of demand functions; however, this work did not discuss the interactions between pricing and the demand in the booking process. To extend and refine the work of Gallego and van Ryzin, Zhao and Zheng used a dynamic pricing model for selling a given stock of a perishable product over a finite time horizon (Zhao and Zheng, 2000), where demand is modelled as a known and decreasing function of the price in a continuous-time stochastic process. In a separate effort, Chatwin allowed the operator to dynamically adjust the price between any of a finite set of allowable prices (Chatwin, 2000), and Feng and Gallego further extended the model by considering the demand intensity as a function of sales to date (Feng and Gallego, 2000). However, none of the above studies specifically treat group travelers separately from individual travelers.

A number of studies on railway passengers and revenue management have been reported, but only a few of them are discussing dynamic pricing. Martin and Nombela applied a gravity model to estimate trip demand for the year 2010 in Spain and then computed the parameters to fit a multinomial logit function (Martín and Nombela, 2007). Bharill used a premium segment of Indian Railways, the Rajdhani Express, to search for revenue management strategies in order to increase average revenue (Bharill and Rangaraj, 2008). In recent years, Sibdari developed a series of pricing policies for a multi-product revenue management problem for the Amtrak Auto Train (Sibdari et al., 2008; Lin and Sibdari, 2009), and You extended the single-fare, multi-leg model, presented by Ciancimino et al., to a two-fare, multi-leg model (You, 2008; Ciancimino et al., 1999). Overall, dynamic pricing on railway passenger operations has not been adequately studied and implemented as compared with the airline industry. In particular, the existing literature has little to report on the impact of group tickets discount to revenue management when dynamic pricing is incorporated, especially in the more price-sensitive HSR context. The motivation of this paper is to evaluate the effectiveness of dynamic pricing in HSR transportation, where the price of group tickets is treated separately and linked to demand function.
3. Model formulation

3.1. Passenger groups of HSR

We define groups with 20 + passengers, which is the case of Chinese Railway system. This model can be extend to deal with other groups (e.g. groups of 4 or more people, and even discounts for couples). Thus, the main characteristics of passenger groups include:

(1) Groups purchase a large number of tickets.
(2) Groups have uniform departure time and travel route.
(3) Groups often use discounted fares since they have a greater chance of being eligible for them than individuals.
(4) Compared with individual passengers, groups generally have a higher show rate and fewer trip cancellations.

3.2. Description of the model and assumptions

Let \( K \) be the total number of seats on a scheduled HSR train, which can be sold at \( I \) different fares. Let \( t \) be the length of time left before departure (\( t = 0 \) is the cut off time to prepare for departure). The booking process starts at \( t = T \) (at this time, all fare classes are available), and it ends when \( t = 0 \). The entire time window \( T \) is divided into many small time periods, and in each period there is at most one booking request (according to the order) in either class \( i (i = 1, 2 \ldots I) \) with probability \( a_i^t \), and no booking request with probability \( 1 - \sum_{i=1}^I a_i^t \).

There are two types of passengers: groups and individuals. Let \( \gamma_i \) denote the price for individuals for class \( i \) ticket. Let \( \gamma_i^g \) denote the discount fare for groups for class \( i \) ticket. Let \( \gamma_i^g(\omega(t, \ k)) \) denote the discount rate, thus \( \gamma_i^g = \gamma_i(\omega(t, \ k)) (0 < \omega(t, \ k) \leq 1) \). There are some lost passengers due to unsatisfactory discount, and let \( \rho(\omega(t, \ k)) \) be the rate of loss associated with discount \( \omega(t, \ k) \). Let continuously differentiable function \( f(\gamma_i^g) \) denote the density function of reservation prices for class \( i \) tickets across the groups. Let \( F(\gamma_i^g) \) denote the cumulative density function of \( f(\gamma_i^g) \). Note \( F(\gamma_i^g) = 1 - F(\gamma_i^g) \), denoting the probability that a randomly chosen group will have a reservation fare higher than \( \gamma_i^g \) and therefore will prefer class \( i \) tickets at price. We assume \( F(\gamma_i^g) \) is strictly increasing as shown in other references (Aksoy-Pierson et al., 2013; Barker, 2005). In any time period, given a price, the probability of a purchase order is \( P(\gamma_i^g, \ y) \) and no order is \( 1 - F(\gamma_i^g) \) (indicating neither purchase nor booking request at all). Price for individuals is fixed, and the probability that a potential individual order appearing at any time period is \( \theta \). The operator's object is to maximize the total expected revenue by offering the optimal set of tickets.

To model the problem outlined above, the following assumptions are made:

1) Only one direction is involved and it does not include multi-station network travels, so there are no passenger additions or subtractions in the middle of the trip.
2) Probability of booking request obeys Gaussian distribution.
3) Overbooking, changing and refunding are not considered.
4) The number of tickets for each order may be described by a Poisson distribution.
5) Purchasing probability of passenger groups may be affected by the ticket fare according to the logit model.
6) Within each time period, we assume that at most one order arrives; that is, the discretization of time is sufficiently fine so that the probability of more than one request is negligible (Talluri and van Ryzin, 1998).

Let \( k \) be the number of remaining tickets at the beginning of any time period, and let \( V(t, k) \) be the maximum expected revenue over time periods \( t = t-1 \ldots 0 \).

Let \( y \) denote the number of tickets to be purchased for each order, where \( y \) follows a Poisson distribution with a mean number \( \lambda \), for \( j = 0, 1, 2 \ldots \) (the number of tickets). The probability mass function of \( y \) is

\[
P(y = j) = \frac{e^{-\lambda} \lambda^j}{j!}
\]

which can be simply written as \( p_j \).

Tickets are discounted if \( y \geq 20 \) according to relevant regulations, and \( F(\gamma_i^g) = 1 - F(\gamma_i^g) \) is the purchase probability of group tickets.

For maximization of the expected revenue, the following two situations must be treated:

1) Tickets are available for either groups or individuals, and groups can purchase discount tickets if \( k \geq 20 \), so
The boundary condition for Equation (1) is \( \forall V(0, k) = 0 \), \( \forall t \), the expected revenue addition is 0 at the last time period, the cut off time for departure.

2) Individuals can only purchase full-fare tickets. If \( k < 20 \), then

\[
V(t, k) = \sum_{i=1}^{l} \alpha_i \left( \sum_{j=1}^{k} P_j \max(\gamma_i) \max[V(t-1, k-j) + \gamma_j, V(t-1, k)] + (1 - \max(\gamma_i)) V(t-1, k) \right)
\]

subject to constraints: \( 20 \leq k \leq K \), \( \sum_{i=1}^{l} \alpha_i \leq 1 \), where \( K \) is the total number of tickets.

The boundary conditions include:

i. \( \forall V(0, k) = 0 \). \( \forall t \), the expected revenue addition is 0 at the departure time cut off time period.

ii. \( \forall V(t, 0) = 0 \). \( \forall t \), the expected revenue addition is 0 if the remaining tickets are sold out in time period \( t \).

3.3. The model for determining the expected revenue of a policy

A numerical experiment, presented in Section 5, was conducted to verify the applicability of the model to get an optimal revenue by dynamic pricing for passenger groups. The optimal control policy as well as the maximum expected revenue can be determined by (1) and (2). The values of total revenue can be determined by the control rule (3) and (4).

\[
g(t, k) = \begin{cases} 
V(t-1, k), & \text{if } V(t-1, k) \geq V(t-1, k-j) + \gamma_j \\
V(t-1, k-j) + \gamma_j, & \text{otherwise}
\end{cases}
\]

(3)

\[
g(t, k) = \begin{cases} 
V(t-1, k), & \text{if } V(t-1, k) \geq V(t-1, k-j) + \gamma_j \\
V(t-1, k-j) + \gamma_j, & \text{otherwise}
\end{cases}
\]

(4)

If the revenue of selling \( j \) tickets is bigger than selling those tickets later, the decision is to sell \( j \) tickets at time \( t \). Otherwise, those tickets will be reserved to sell later. Equation (3) is for individuals and Equation (4) is for groups.
Given the boundary condition $V(0, k) = 0$, for each time period $t$, the expected revenue $V(t, k)$ is evaluated based on the control rules of the previous stage $g(t-1, k)$ or the combination of $g(t-1, k)$ and $g(t-1, k)$ as shown in (3) and (4). The whole determination procedure of the expected revenue of a policy is illustrated in Fig. 1.

An inverse dynamic programming method is used to solve the model. The boundary condition is $V(0, k) = 0$. This algorithm is described as following:

**Step 1:** set $t = 1$;
**Step 2:** if $t < T$, set $k = 1$; Else, go to step 5;
**Step 3:** if $k < 20$, calculate the value of $g(t, k)$ through control formula 3;
    if $K > 20$, calculate the value of $g(t, k)$ through control formula 4;
    if $k > K$, $t = t + 1$, go to step 2;
**Step 4:** set $k = k + 1$, go to step 3;
**Step 5:** end

### 4. Numerical experiment

In this section, a real-world example of the HSR transport from Beijing to Shanghai, to show the effectiveness of the discussed model in this paper. There are 3 types of seats: 791 Second-Class seats, 186 First-Class seats and 28 Business seats, resulting in a proportion of classes as $[0.787, 0.185, 0.028]$. Let class $I$ denote the Second-Class, class $II$ denote the First-Class and class $III$ denote the Business, the operator has to sell 1005 tickets within $T = 2000$ time periods with price $g_1 = 553$, $g_2 = 993$, $g_3 = 1748$. In the real data, only two levels of price variation for passenger groups, full fare and discount fare, were used.

We have investigated the optimization problem by using (MATLAB, R2010b) to solve the mathematical formulations. Let denote the mean and $\sigma$ denote the variance of demand probabilities. By analyzing the travel survey data we obtained following parameters shown in Table 1 for different classes.

We plotted a diagram of the demand distribution (obtained from travel survey data) and the corresponding booking request probabilities for $T$. Those passengers who purchase class $III$ are not sensitive to price. Those passengers who purchase class tickets are price sensitive. *Price sensitive means that passengers are likely to reject the ticket if the price goes up; price insensitive means they are not likely to do so.*

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**Fig. 1.** Determination of the expected revenue of a policy.
The left side subgraph of Fig. 2 shows that peak demand of class III is close to the departure time while the class I demand change is just the opposite to that of class III. We cumulated the demand data and normalized the distribution by using the parameters in Table 1 to obtain the booking request probabilities depicted as the right side subgraph of Fig. 2.

4.1. Effect of discount on purchasing

Let denote the minimum fare of group tickets and be the corresponding purchase probability, a logit relationship as defined in Section 3.2. Accordingly, $F(x) = \frac{1 + c(1 + e^{(x-a)/b})}{1 + c(1 + e^{(x-a)/b})}$, where $b$ is the price-sensitive factor, and $a$ is the adjustment coefficient and $c$ is the constant to normalize $F(x)$. To simulate market conditions, we calibrated those coefficients by using the data from the Chinese Railway Corporation and obtained $a = [-4.8, -4.5, -4.9], b = [0.016, 0.009, 0.005]$, and $c = [0.25, 0.15, 0.05]$ (for which details of the estimation process is presented elsewhere (Anderson, 1988)).

As show in Fig. 3, class I is the cheapest one and number of groups who prefer to buy class I seat will decline sharply. Class III is the expensive one and number of groups who prefer to buy class III seat will decline slowly. It also demonstrate that class III is not sensitive to price and class I is sensitive to price.

The left subgraph of Fig. 3 shows changes of the observed cumulative purchase probability as affected by prices in the range [0, 1800]. Obviously, the purchase probability of group travelers is decreasing with the price increase. At lower discount price the majority of groups will choose HSR; but if $\omega$ is higher, ticket prices approaching the original price, some passengers may abandon HSR for other modes of travel, which leads to a larger rate of loss as depicted in the right side subgraph of Fig. 3.

4.2. Effect of discount on total expected revenue

As explained in Section 5, the seats of all classes have the same discount rate in the range [0, 1]. Accordingly, the trend of total revenue change is shown in Fig. 4.

It can be observed from Fig. 4 that although a lower discount price leads to a higher demand probability of group travelers (as discussed before), it tends to reduce total revenue. In the range [0.1, 0.74] the total revenue is increasing with discount. If $\omega = 0.74$, we will get the highest total revenue, $V = 1,069,546.6$. However, if $\omega$ continues to increase and approaches the

<table>
<thead>
<tr>
<th>Class</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.8T</td>
<td>0.6T</td>
<td>0.15T</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.3T</td>
<td>0.4T</td>
<td>0.1T</td>
</tr>
</tbody>
</table>

Table 1 Parameters of demand probabilities.
Fig. 3. Purchase probability and rate of loss.

Fig. 4. Total revenue due to discount.
original price in the range \([0.74, 1]\), the total revenue decreases. We denote the optimal discount rate to reach the maximal total revenue as \(u_0\).

The total revenue \(V\) increases with time period \(t\) and sales of the remaining tickets \(k\), as shown in Fig. 5. The total expected revenue increases the earlier the tickets are sold. In another words, the earlier the railway operator begins selling tickets, the higher profit. Note that if the discount rates for each class were not equal, then we should use \([\omega_1, \omega_2, \omega_3]\) to represent the discount rates of different ticket classes.

4.3. The effect of demand-supply ratio on optimal discount

In most cases, supply (available seats) is greater than the travel demand (ordered seats) except for the holiday travels. Let \(\pi\) denote the proportion of order seats to seat supply and \(\pi\) will affect \(T\) value. Within each time period, we assume that at most one order arrive; that is, the discretization of time is sufficiently fine so that the probability of more than one request is negligible. If \(\pi\) is larger, the \(T\) value will be larger accordingly. The detail information can be found in this paper (Talluri and van Ryzin, 1998). Table 2 shows the optimal discount \(u_0\) and total revenue for different \(\pi\), where \(\pi\) changes from \(\pi < 1\) (more supply) to \(\pi > 1\) (less supply).

The left part of Fig. 6 shows that the bigger \(\pi\), the higher tickets price (lower discount). With the same demand rate, discount of class III is higher than that of class I. If \(\pi\) is bigger than 2, all classes will be sold in full fare.

It can be observed from Table 2 that the optimal discount rates change with the demand over supply ratios (\(\pi\)) and generally the total revenue grows with an increasing demand (as shown in the right side subgraph of Fig. 6), indicating the advantage on the supply side during the busy season and less pressure on the railroad corporation to offer discount tickets. Furthermore, it shows that there is no need to offer a discount (i.e., \(\omega = 1\)) when \(\pi\) reaches 0.7 for Class I, whereas for Class III the required \(\pi\) value for no discount sale is at 2. This is only likely to happen in major holidays. This feature is depicted in the left side subgraph of Fig. 6, indicating the price competitiveness of coach class over the business class.

**Table 2**

<table>
<thead>
<tr>
<th>(\pi)</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>class I</td>
<td>0.55</td>
<td>0.65</td>
<td>0.72</td>
<td>0.86</td>
<td>0.89</td>
<td>0.94</td>
<td>0.94</td>
<td>0.96</td>
</tr>
<tr>
<td>class II</td>
<td>0.55</td>
<td>0.64</td>
<td>0.69</td>
<td>0.81</td>
<td>0.85</td>
<td>0.85</td>
<td>0.90</td>
<td>0.96</td>
</tr>
<tr>
<td>Total revenue</td>
<td>482,401.4</td>
<td>533,476.1</td>
<td>673,845.5</td>
<td>797,522.9</td>
<td>847,414.8</td>
<td>893,107.0</td>
<td>933,293.6</td>
<td>963,180.3</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0.9</td>
<td>1</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
<td>1.4</td>
<td>1.5</td>
<td>1.6</td>
</tr>
<tr>
<td>class I</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>class II</td>
<td>0.99</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>class III</td>
<td>0.71</td>
<td>0.74</td>
<td>0.76</td>
<td>0.79</td>
<td>0.81</td>
<td>0.83</td>
<td>0.86</td>
<td>0.89</td>
</tr>
<tr>
<td>Total revenue</td>
<td>991,013.7</td>
<td>1,029,874.2</td>
<td>1,045,028.6</td>
<td>1,070,067.8</td>
<td>1,105,225.4</td>
<td>1,135,281.9</td>
<td>1,163,205.2</td>
<td>1,188,909.5</td>
</tr>
<tr>
<td>(\pi)</td>
<td>1.7</td>
<td>1.8</td>
<td>1.9</td>
<td>2</td>
<td>2.1</td>
<td>2.2</td>
<td>2.3</td>
<td>2.4</td>
</tr>
<tr>
<td>class I</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>class II</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>class III</td>
<td>0.92</td>
<td>0.95</td>
<td>0.99</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total revenue</td>
<td>1,211,343.1</td>
<td>1,229,044.0</td>
<td>1,242,919.7</td>
<td>1,254,181.3</td>
<td>1,263,406.2</td>
<td>1,271,649.6</td>
<td>1,279,227.8</td>
<td>1,286,431.0</td>
</tr>
</tbody>
</table>

**Fig. 5.** Total revenue \(V\) as affected by \(k\) and \(t\).
5. Conclusion

The goal of this paper is to build a model to find the optimal discount rate for passenger groups in order to maximize the expected total revenue in high speed rail transportation. **Our result** is helpful for Chinese Railway Corporation to increase profit by adopt flexible discount for groups. The discount tickets will attract more price-sensitive travels during off-season.

We have developed a piecewise function for dynamic pricing by differentiating the group and individual travelers. Compared with existing static models, our logit-model based dynamic pricing approach uses different order size for each seat class to represent the varying demand. Numerical experiment suggests that proposed method can calculate the optimal discount rate which can maximize the total expected revenue.

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Appendix A. Properties analyzing

When \( k \geq 20 \), let

\[
H_1(t, k; \gamma^i) = \max_{u(t,k)} \{ F(\gamma^i) \max [(V(t-1, k-j) + \gamma^i j), V(t-1, k)] + (1 - F(\gamma^i)) V(t-1, k) \}
\]

represents the expected revenue of a group passengers.

When \( k < 20 \), let

\[
H_2(t, k; \gamma^i) = F(\gamma^i) \max [(V(t-1, k-j) + \gamma^i j), V(t-1, k)] + (1 - F(\gamma^i)) V(t-1, k)
\]

represents the expected revenue of an individual passenger.
So \( k \geq 20 \)

\[
H_1(t, k, \gamma_i) = \max_{\omega(t,k)} \{F(\gamma_i') \max \left[ (V(t-1, k-j) + \gamma_j'), (V(t-1, k) + (1 - F(\gamma_i'))V(t-1, k) \right] \\
H_2(t, k, \gamma_i) = \max_{\omega(t,k)} \{F(\gamma_i') \max (V(t-1, k-j) + \gamma_j', V(t-1, k)) - V(t-1, k) + V(t-1, k) \}
\]

\( k < 20 \)

\[
H_1(t, k, \gamma_i) = F(\gamma_i') \max (V(t-1, k-j) + \gamma_j', V(t-1, k)) - V(t-1, k) + V(t-1, k) \\
H_2(t, k, \gamma_i) = F(\gamma_i') \max (V(t-1, k-j) + \gamma_j' - V(t-1, k), 0] + V(t-1, k)
\]

Let \( A = V(t-1, k-j) - V(t-1, k) \)

\[
H_1(t, k, \gamma_i) = \max_{\omega(t,k)} \{F(\gamma_i') \max [A + \gamma_j', 0] \} + V(t-1, k) \\
H_2(t, k, \gamma_i) = F(\gamma_i') \max [A + \gamma_j', 0] + V(t-1, k) = V(t-1, k)
\]

If \( A + \gamma_j' \leq 0 \), that \( A \leq -\gamma_j' \)

\[
H_1(t, k, \gamma_i) = \max_{\omega(t,k)} \{F(\gamma_i') \times 0 + V(t-1, k) \} = V(t-1, k) \\
H_2(t, k, \gamma_i) = F(\gamma_i') \max [A + \gamma_j', 0] + V(t-1, k) = V(t-1, k)
\]

If \( A + \gamma_j' > 0 \), that \( A > -\gamma_j' \)

\[
H_1(t, k, \gamma_i) = \max_{\omega(t,k)} \{F(\gamma_i') (A + \gamma_j') + V(t-1, k) \} \\
H_2(t, k, \gamma_i) = F(\gamma_i') \max [A + \gamma_j', 0] + V(t-1, k) = F(\gamma_i')(A + \gamma_j') + V(t-1, k)
\]

Let \( g_1(A) = \max_{\omega(t,k)} \{F(\gamma_i') (A + \gamma_j') \}, g_2(A) = \max_{\omega(t,k)} F(\gamma_i')(A + \gamma_j') \)

\[
\gamma_i^*(A) = \arg \max_{\omega(t,k)} \{F(\gamma_i') (A + \gamma_j') \}, \gamma_i^*(A) = \arg \max_{\omega(t,k)} F(\gamma_i')(A + \gamma_j')
\]

**Lemma 1**: If \( A > -\gamma_j' \),

1) \( \gamma_i^*(A) \) and \( \gamma_i^*(A) \) increase with the increase of \( A \)
2) \( g_1(A) \) and increase with the decrease of \( A \)

**Proof**:

1) Without loss of generality, hypothesize \( A > B \). \( \gamma_i^*(A) \) is the optimal price of \( g_1(A) \). \( \gamma_i^*(A) \) is the optimal price of \( g_2(A) \). \( \gamma_i^*(B) \) is the optimal price of \( g_1(B) \). \( \gamma_i^*(B) \) is the optimal price of \( g_2(B) \). According to the optimality of \( \gamma_i^*(A) \):

\[
F(\gamma_i^*(A)) (A + \gamma_i^*(A)j) > F(\gamma_i^*(B)) (A + \gamma_i^*(B)j)
\]

According to the optimality of \( \gamma_i^*(B) \):

\[
F(\gamma_i^*(B)) (B + \gamma_i^*(B)j) > F(\gamma_i^*(A)) (B + \gamma_i^*(A)j)
\]

Using apagoge:

Hypothesize \( \gamma_i^*(B) > \gamma_i^*(A) \), so \( F(\gamma_i^*(B)) < F(\gamma_i^*(A)) \)
\[0 > F(\gamma_i^*(A))(B + \gamma_i^*(A)j) - F(\gamma_i^*(B))(B + \gamma_i^*(B)j) = F(\gamma_i^*(A))(A + \gamma_i^*(A)j) - F(\gamma_i^*(B))(A + \gamma_i^*(B)j) + (B - A)F(\gamma_i^*(B)) - F(\gamma_i^*(A)) \]

According to the optimality of \(\gamma_i^*(A)\),
\[F(\gamma_i^*(A))(A + \gamma_i^*(A)j) - F(\gamma_i^*(B))(A + \gamma_i^*(B)j) > 0\]

From hypothesis: \((B - A)F(\gamma_i^*(B)) - F(\gamma_i^*(A)) > 0\)
\[.0 > F(\gamma_i^*(A))(B + \gamma_i^*(A)j) - F(\gamma_i^*(B))(B + \gamma_i^*(B)j) > 0\]

Hypothesis contradiction. ∴ When \(A > B\), \(\gamma_i^*(A) > \gamma_i^*(B)\)

Hypothesize \(\gamma_i^*(B) > \gamma_i^*(A)\),
\[0 > F(\gamma_i)(A + \gamma_i^*(B)j) - F(\gamma_i)(A + \gamma_i^*(A)j) = F(\gamma_i)A + F(\gamma_i)\gamma_i^*(B)j - F(\gamma_i)A - F(\gamma_i)\gamma_i^*(A)j = F(\gamma_i)\gamma_i^*(B)j - F(\gamma_i)\gamma_i^*(A)j = F(\gamma_i)j(\gamma_i^*(B) - \gamma_i^*(A))\]

From hypothesis, \(F(\gamma_i)(A + \gamma_i^*(B)j) - F(\gamma_i)(A + \gamma_i^*(A)j) > 0\) hypothesis contradiction.
∴ When \(A > B\), \(\gamma_i^*(A) > \gamma_i^*(B)\)

2) Hypothesize \(B > A\)

\[g_1(A) = F(\gamma_i^*(A))(A + \gamma_i^*(A)j) < F(\gamma_i^*(A))(B + \gamma_i^*(A)j) < F(\gamma_i^*(B))(B + \gamma_i^*(B)j) = g_1(B)\]

\[g_2(A) = F(\gamma_i)(B + \gamma_i^*(B)j) < F(\gamma_i)(B + \gamma_i^*(A)j) < F(\gamma_i)(B + \gamma_i^*(B)j) = g_2(B)\]

When \(B > A\), \(g_1(A) < g_1(B)\), \(g_2(A) < g_2(B)\)

References